Stresses due to Rotation

Chapter 19

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19.1. INTRODUCTION

The bodies like rings, circular discs, cylinders, flywheels, etc. invariably rotate at high speeds and due to rotation they are subjected to large magnitudes of centrifugal forces. The stresses caused by these forces are distributed symmetrically about their axes of rotation. In this chapter we shall study these stresses for machine members of the rotating type; assuming the density of material to be uniform throughout.

19.2. ROTATING RING

Fig. 19.1 (a) shows the thin ring rotating about its centre of gravity at 0.

Let,

- \( r \) = Mean radius of the ring, m,
- \( t \) = Thickness of the ring, m,
- \( \rho \) = Density of the material of the ring, kg/m\(^3\),
- \( \omega \) = Angular speed of the ring, radian/sec.,
- \( F_c \) = Centrifugal force, and
- \( \sigma_c \) = Circumferential or hoop stress.
As a result of rotation each and every element of the ring like $LMPQ$ will experience centrifugal (or inertia) force $dF_c$ which will tend to expand the ring radially outwards. This will in turn induce the circumferential (or hoop stress) $\sigma_c$ in the ring which will be tensile in nature. For evaluating this stress following assumptions are made:

(i) The circumferential stress on the area of cross-section of the ring is uniform.
(ii) The dimensions of the cross-section of the ring are small as compared to its mean radius.
(iii) The constraining effect of spokes is negligible.

Now, volume of small element (Fig. 19.1(b)) $LMPQ$ per unit length $= r\,d\theta\,t$

Hence centrifugal force acting on the element,

$$dF_c = \rho \, r\,d\theta\,t \cdot \omega^2 r \quad (\because F_c = m \, \omega^2 r)$$

Vertical component of $dF_c$

$$dF_c = dF_c \, \sin \theta$$

$$= \rho \, r \, d\theta \, t \, \omega^2 r \, \sin \theta$$

Horizontal component of $dF_c$ will be cancelled when we consider another small element $L'M'P'Q'$ in II quadrant at an angle $\theta$, but the vertical component of $dF_c$ will be added.

Total vertical component or bursting force across the horizontal diameter $XX$

$$= \int_0^\frac{\pi}{2} dF_c \, \sin \theta = \int_0^\frac{\pi}{2} \rho \, r \, d\theta \, t \, \omega^2 r \, \sin \theta$$

$$= \rho \, r^2 \, t \, \int_0^\frac{\pi}{2} \sin \theta \, d\theta$$

$$= \rho \, r^2 \, t \, \left[ -\cos \theta \right]_0^{\frac{\pi}{2}} = 2 \rho \, r^2 \, t$$

Now, total resisting force

$$= 2 \sigma_c \, t \, l$$

The ring will be in equilibrium, when:

Total bursting force = Total resisting force

$$2\rho \, \omega^2 \, r \, t = 2 \sigma_c \, t \, l$$

$$\therefore \quad \sigma_c = \rho \, \omega^2 \, r$$

But,

$$v = \omega \, r$$

(where, $v =$ linear velocity of the ring)

$$\therefore \quad \sigma_c = \rho \, v^2$$
Example 19-1. A wheel 800 mm in diameter has a thin rim. If density is 7700 kg/m³ and Young’s modulus is 200 GN/m², calculate:

(i) How many revolutions per minute may it make without the hoop stress exceeding 130 MN/m²?
(ii) Change in diameter.

Neglect the effect of spokes.

Solution. Radius of the wheel, \( r = \frac{800}{2} = 400 \text{ mm} = 0.4 \text{ m} \)

Density of material, \( \rho = 7700 \text{ kg/m}^3 \)

Young’s modulus, \( E = 200 \text{ GN/m}^2 \)

Hoop stress, \( \sigma_c = 130 \text{ MN/m}^2 \).

(i) Number of revolutions, \( N \):

We know that \( \sigma_c = \rho \omega^2 r^2 \)

\( 130 \times 10^6 = 7700 \times \omega^2 \times 0.4^2 \)

\( \therefore \omega = 324.8 \text{ rad/sec} \)

But, \( \omega = \frac{2\pi N}{60} = 324.8 \)

\( \therefore N = \frac{324.8 \times 60}{2\pi} = 31016 \text{ r.p.m} \) (Ans.)

(ii) Change in diameter, \( \delta d \):

Circumferential strain, \( e_c = \frac{\sigma_c}{E} \)

or, \( \frac{\delta d}{d} = \frac{130 \times 10^6}{200 \times 10^9} \)

\( \therefore \delta d = 0.8 \times \frac{130 \times 10^6}{200 \times 10^9} = 5.2 \times 10^{-4} \text{ m} = 0.52 \text{ mm} \)

Hence, \( \delta d = 0.52 \text{ mm} \) (Ans.)

Example 19-2. Fig. 19-2 shows a built-up ring. If the ring rotates at 2000 r.p.m., find the stresses set up in steel and copper rings.
Assume: For steel, \( E = 200 \text{ GN/m}^2; \rho = 7800 \text{ kg/m}^3 \)

For copper, \( E = 100 \text{ GN/m}^2; \rho = 8900 \text{ kg/m}^3 \)

**Solution.** Given: \( E_s = 200 \text{ GN/m}^2; \rho_s = 7800 \text{ kg/m}^3 \)
\( E_{cu} = 100 \text{ GN/m}^2; \rho_{cu} = 8900 \text{ kg/m}^3 \)

Speed, \( N = 2000 \text{ r.p.m.} \)

**Stresses in steel and copper rings:**

Let, \( p = \) Contact pressure between the steel and copper rings, MN/m²

Circumferential stress in the steel ring,
\[
\sigma_c = \frac{pd}{2t} = \frac{p \times 720}{2 \times 18} = 20 \ p \text{ MN/m}^2 \text{ (tensile)}
\]

Circumferential stress in the copper ring,
\[
\sigma_{cu} = \frac{pd}{2t} = \frac{p \times 720}{2 \times 18} = 20 \ p \text{ MN/m}^2 \text{ (comp.)}
\]

Circumferential stresses due to rotation in the rings:

**Steel ring:**
\[
\sigma_c = \rho \nu^2
\]

where,
\[
\nu = \omega r = 2\pi N \frac{R}{60}
\]

\[
\rho = 2\pi \times \frac{2000}{60} \times [(360 + 9) \times 10^{-3}] = 77.28 \text{ m/s}
\]

\[
\sigma_c' = 7800 \times (77.28)^2 \times 10^{-6} \text{ MN/m}^2 = 46.58 \text{ MN/m}^2 \text{ (tensile)}
\]

**Copper ring:**
\[
\sigma_{cu}' = \rho_{cu} \nu^2
\]

where,
\[
\nu = \omega r = 2 \pi N r_{cu}
\]

\[
= 2\pi \times \frac{2000}{60} \times [(360 - 9) \times 10^{-3}] = 73.51 \text{ m/s}
\]

\[
\sigma_{cu}' = 8900 \times (73.51)^2 \times 10^{-6} \text{ MN/m}^2 = 48.1 \text{ MN/m}^2 \text{ (tensile)}
\]

**Total stress in steel**
\[
(\sigma_c + \sigma_c') = 20 \ p + 46.58
\]

**Total stress in copper**
\[
(\sigma_{cu}' - \sigma_{cu}) = 48.1 - 20 \ p
\]

Now, circumferential strain in steel = Circumferential strain in copper
\[
\frac{(20p + 46.58) \times 10^6}{200 \times 10^2} = \frac{(48.1 - 20p) \times 10^6}{100 \times 10^9}
\]

or,
\[
20p + 46.58 = 2(48.1 - 20p)
\]

\[
p = 0.827 \text{ MN/m}^2
\]

**Total stress in steel ring**
\[
20 \times 0.827 + 46.58 = 63.12 \text{ MN/m}^2 \text{ (Ans.)}
\]

And, total stress in copper ring
\[
48.1 - 20 \times 0.827 = 31.56 \text{ MN/m}^2 \text{ (Ans.)}
\]
19.3. ROTATING THIN DISC

Fig. 19.3 (a) shows a circular disc of inner radius $r_1$ and outer radius $r_2$ rotating about its axis.

Let us assume that the disc is of uniform thickness and that the thickness is so small compared with its diameter that there is no variation of stress along the thickness. At the free flat surfaces there can be no stress normal to these faces and there can be no shear stress on or perpendicular to these faces. Thus the direction of axis is the direction of zero principal stress. The displacement of any point due to strain must be radial. The radial and circumferential stresses, therefore, represent the principal stresses.

Consider an element $ABCD$ of the disc, at a radius $r$, subtending an angle $d\theta$ at the centre, and of radial width $dr$.

Let,

\[
\sigma_r = \text{Stress on the face } CD,
\]
\[
\sigma_r + d\sigma_r = \text{Stress on the face } AB,
\]
\[
\sigma_c = \text{Stress on face } BC,
\]
\[
\sigma_c = \text{Stress on face } AD.
\]

On the flat faces of the disc; there is no normal stress, and hence there is free strain in the direction of the axis.

Volume of the element

\[ABCD = rd\theta . dr . t\] (where, $t$ = thickness of the disc)

Radial force on the element $ABCD$ due to rotation

\[= \rho \ r d\theta . dr . t . \omega^2 \ r = \rho \ r d\theta . dr . t \omega^2 \ r^2\]

Force on face $AB$ (outward)

\[= (r + dr) d\theta . t (\sigma_r + d\sigma_r)\]

Force on $CD$ (inward)

\[= r . d\theta . t . \sigma_r\]

Force on faces $BC$ and $AD = \sigma_c . t . dr$

[Forces acting on the element are shown in Fig. 19.3 (b)].
Resolving the forces in the radial direction and considering the equilibrium of the forces, we get

\[ r d\theta \cdot t \cdot \sigma_r + 2 \sigma_c \cdot t \cdot dr \cdot \sin \left( \frac{d\theta}{2} \right) = (\sigma_r + d\sigma_r) \cdot (r + dr) \cdot d\theta \cdot t + \rho \cdot d\theta \cdot dr \cdot t \cdot \omega^2 r^2 \]

Since \( d\theta \) is very small, \( \sin \frac{d\theta}{2} \approx \frac{d\theta}{2} \)

Now, \( t \cdot d\theta \) is common on both the sides, the above expression on simplification becomes

\[ \sigma_r \cdot r + \sigma_c \cdot dr = \sigma_r \cdot r + \sigma_r \cdot dr + r \cdot d\sigma_r + d\sigma_r \cdot dr + \rho \cdot \omega^2 r^2 dr \]

Neglecting the term \( d\sigma_r \cdot dr \) and on further simplification the expression becomes

\[ \sigma_c \cdot dr = \sigma_r \cdot dr + r \cdot d\sigma_r + \rho \cdot \omega^2 r^2 dr \]

Dividing both sides by \( dr \), we get

\[ \sigma_c = \sigma_r + r \cdot \frac{d\sigma_r}{dr} + \rho \cdot \omega^2 r^2 \]

or,

\[ \sigma_c - \sigma_r = r \cdot \frac{d\sigma_r}{dr} + \rho \cdot \omega^2 r^2 \]

...(i)

Let us now consider circumferential and radial strains.

When the disc is rotating at high speed let \( r \) become \( (r + u) \) and \( (r + dr) \) become \( (r + dr + du) \)

Circumferential strain,

\[ e_c = \frac{2\pi (r + u) - 2\pi r}{2\pi r} = \frac{u}{r} \]

Radial strain,

\[ e_r = \frac{(r + dr + du) - (r + dr)}{dr} = \frac{du}{dr} \]

Also,

\[ e_c = \frac{1}{E} \left( \sigma_c - \frac{\sigma_r}{m} \right) = \frac{u}{r} \]

\[ u = \frac{r}{E} \left( \sigma_c - \frac{\sigma_r}{m} \right) \]

\[ \therefore \]

Differentiate (ii)
\[ \sigma_r = \frac{1}{E} \left( \sigma_r - \sigma_c \right) = \frac{du}{dr} \]  

(where, \( \frac{1}{m} = \text{Poisson's ratio} \) 

Differentiating equation (ii) w.r.t \( r \), we get

\[ \frac{du}{dr} = \frac{r}{E} \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] + \frac{1}{E} \left( \sigma_c - \sigma_r \right) \]  

...(iii)

Comparing equations (iii) and (iv), we get

\[ \frac{1}{E} \left( \sigma_r - \sigma_c \right) \frac{r}{m} \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] + \frac{1}{E} \left( \sigma_c - \sigma_r \right) = \frac{r}{E} \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] \]

...(iv)

\[ 1 + \frac{1}{m} \] \( \frac{d\sigma_r}{dr} = \frac{r}{E} \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] \)

...(v)

Substituting eqn (iv) in eqn. (v), we get

\[ \left( 1 + \frac{1}{m} \right) \left[ -r \frac{d\sigma_r}{dr} - \rho \omega^2 r^2 \right] = r \left( \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right) \]

On simplification, we get

\[ \frac{d}{dr} \left( \sigma_r + \sigma_c \right) + \left( 1 + \frac{1}{m} \right) \rho \omega^2 r = 0 \]  

...(vi)

Integrating equation (vi), we get

\[ \sigma_r + \sigma_c + \left( 1 + \frac{1}{m} \right) \frac{\rho \omega^2 r^2}{2} = C_1 \]

(\text{where, } C_1 = \text{constant of integration})

\[ \sigma_c = C_1 - \sigma_r - \left( 1 + \frac{1}{m} \right) \frac{\rho \omega^2 r^2}{2} \]  

Substituting in eqn. (i), we get

\[ C_1 - \sigma_r - \left( 1 + \frac{1}{m} \right) \frac{\rho \omega^2 r^2}{2} - \sigma_r = r \frac{d\sigma_c}{dr} + \rho \omega^2 r^2 \]

\[ 2\sigma_r + r \frac{d\sigma_r}{dr} = C_1 - \rho \omega^2 r^2 \left[ 1 + \left( 1 + \frac{1}{m} \right) \frac{\rho \omega^2 r^2}{2} \right] \]

\[ 2\sigma_r + r \frac{d\sigma_r}{dr} = C_1 - \left( 1 + \frac{1}{m} \right) \rho \omega^2 r^2 \]

\[ 2\sigma_r + r \frac{d\sigma_r}{dr} = C_1 - \left( \frac{3 + \frac{1}{m}}{2} \right) \rho \omega^2 r^2 \]

Multiplying both sides by \( r \), we get

\[ 2r\sigma_r + r^2 \frac{d\sigma_r}{dr} = C_1 r - \left( \frac{3 + \frac{1}{m}}{2} \right) \rho \omega^2 r^3 \]

or,

\[ \frac{d}{dr} \left( r^2 \sigma_r \right) = C_1 r - \left( \frac{3 + \frac{1}{m}}{2} \right) \rho \omega^2 r^3 \]
Integrating both sides, we get
\[
r^2 \sigma_r = C_1 \cdot \frac{r^2}{2} - \left( \frac{3 + \frac{1}{m}}{2} \right) \rho \omega^2 \frac{r^4}{4} + C_2.
\]

or,
\[
\sigma_r = \frac{C_1}{2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2 + \frac{C_2}{r^2}
\]

or,
\[
\sigma_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]

(where, \(C_2\) = constant of integration)

Substituting eqn. (19-3) in equation (vii), we get
\[
\left[ \frac{C_1}{2} - \frac{3 + \frac{1}{m}}{8} \right] \rho \cdot \omega^2 r^2 + \frac{C_2}{r^2} + \sigma_c + \left( \frac{1 + \frac{1}{m}}{2} \right) \rho \cdot \omega^2 r^2 = C_1
\]

\[
\sigma_c = \frac{C_1}{2} - \frac{C_2}{r^2} + \rho \cdot \omega^2 r^2 \left( \frac{3 + \frac{1}{m}}{8} \right) - \left( \frac{1 + \frac{1}{m}}{2} \right)
\]

\[
\sigma_c = \frac{C_1}{2} - \frac{C_2}{r^2} - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]

(19.4)

The constants can be evaluated by using the boundary conditions.

Hence eqns. (19-3) and (19-4) give the expressions for radial and circumferential stresses in the disc.

**Case I. Solid disc:**

At the centre \(r = 0\), and stresses cannot be infinite at the centre of the disc, therefore, \(C_2 = 0\). Expressions for stresses will now be:

\[
\sigma_r = \frac{C_1}{2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]

\[
\sigma_c = \frac{C_1}{2} - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]

At the outer radius, \(r = r_2\), \(\sigma_r = 0\)

\[
\therefore \quad C_1 = \frac{3 + \frac{1}{m}}{4} \rho \omega^2 r_2^2
\]

\[
\therefore \quad \sigma_r = \frac{3 + \frac{1}{m}}{8} \rho \omega^2 (r_2^2 - r^2)
\]

and,
\[
\sigma_c = \frac{3 + \frac{1}{m}}{8} \rho \omega^2 r_2^2 - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]

(19.5)
or,
\[
\sigma_c = \frac{\rho \omega^2}{8} \left[ \left( 3 + \frac{1}{m} \right) r_2^2 - \left( 1 + \frac{3}{m} \right) r^2 \right] \quad \ldots (19-6)
\]

At \( r = r_2 \),
\[
\sigma_c = \left( \frac{1 - \frac{1}{m}}{4} \right) \rho \omega^2 r_2^2 \quad \ldots (19-7)
\]

At \( r = 0 \) the values of \( \sigma_r \) and \( \sigma_c \) are maximum; these are:
\[
(\sigma_r)_{\text{max}} = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_2^2 \quad \ldots (19-8)
\]
\[
(\sigma_c)_{\text{max}} = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_2^2 \quad \ldots (19-9)
\]

Fig. 19.4 shows variations of \( \sigma_c \) and \( \sigma_r \) in a solid disc.

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**Case II. Hollow disc:**

From eqns. (19-3) and (19-4), we have
\[
\sigma_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]
\[
\sigma_c = \frac{C_1}{2} - \frac{C_2}{r^2} - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]

Boundary conditions are:
At \( r = r_1 \), \( \sigma_r = 0 \) and, at \( r = r_2 \), \( \sigma_r = 0 \)
\[
0 = \frac{C_1}{2} + \frac{C_2}{r_1^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_1^2 \quad \ldots (i)
\]
\[
0 = \frac{C_1}{2} + \frac{C_2}{r_2^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_2^2 \quad \ldots (ii)
\]

From these equations, we get
\[
\frac{C_2}{r_1^2} - \frac{C_2}{r_2^2} = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_1^2 - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_2^2
\]
\[
C_2 \left[ \frac{r^2 - r_1^2}{r_1^2 r_2^2} \right] = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 (r_1^2 - r_2^2) \quad \therefore \quad C_2 = -\left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_1^2 r_2^2
\]

Substituting the value of \( C_2 \) in eqn (i), we get

\[
0 = C_1 \left( 2 \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_2^2 - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r_1^2 \right)
\]

\[
\therefore \quad C_1 = \left( \frac{3 + \frac{1}{m}}{4} \right) \rho \omega^2 (r_1^2 + r_2^2)
\]

\[
\therefore \quad \sigma_r = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 (r_1^2 + r_2^2) - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \frac{r_1^2 r_2^2}{r^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]

or,

\[
\sigma_r = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \left[ r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2 \right] \quad \ldots(19-10)
\]

and,

\[
\sigma_c = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 (r_1^2 + r_2^2) + \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \frac{r_1^2 r_2^2}{r^2} - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]

\[
\sigma_c = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \left[ r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left( \frac{1 + \frac{3}{m}}{3 + \frac{1}{m}} \right) r_1^2 \right] \quad \ldots(19-11)
\]

Inspection of eqn. (19-11) shows that \( \sigma_c \) goes on increasing as \( r \) decreases and hence \( \sigma_c \) is maximum when \( r = r_1 \).

\[
\therefore \quad (\sigma_c)_{\text{max}} = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \left[ r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left( \frac{1 + \frac{3}{m}}{3 + \frac{1}{m}} \right) r_1^2 \right]
\]

Car wheel with tyre. When brakes are applied they exert torque on car wheels.
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\[(\sigma_c)_{\text{max}} = \left(3 + \frac{1}{m}\right)\rho\omega^2 \left[\frac{1}{r_2^2} + \frac{1}{3 + \frac{1}{m}} \frac{r_1^2}{r_2^2}\right] \quad \text{...(19-12)}\]

For \(\sigma_r\) to be maximum, \(\frac{d\sigma_r}{dr} = 0\)

\[\frac{d}{dr} \left[\left(3 + \frac{1}{m}\right)\rho\omega^2 \left(\frac{r_1^2 r_2^2}{r_2^2} - \frac{r_1^2 r_2^2}{r_1^2} - r^2\right)\right] = 0\]

\[\frac{2r_1^2 r_2^2}{r_3^2} - 2r = 0\]

\[r^4 = r_1^2 r_2^2, \quad \text{or}, \quad r = \sqrt{r_1 r_2} \quad \text{...(19-13)}\]

\[\sigma_r \rightarrow \left(\frac{3 + \frac{1}{m}}{8}\right)\rho\omega^2 \left[(r_2 - r_1)^2\right] \quad \text{...(19-14)}\]

...substituting the value of \(r\) in equation (19-10)

The variations of \(\sigma_c\) and \(\sigma_r\), with \(r\) are shown in Fig. 19.5.

\[\sigma_c\] and \(\sigma_r\) in a hollow disc.

**Hollow disc with a pin hole at the centre**

Here \(r_1 \to 0\), then from eqns. (19-12) and (19-13)

\[\sigma_c \to \left(3 + \frac{1}{m}\right)\rho\omega^2 r_2^2 \quad \text{...(19-15)}\]

and,

\[\sigma_r \to \left(3 + \frac{1}{m}\right)\rho\omega^2 r_2^2 \quad \text{...(19-16)}\]

Comparing the eqn. (19-15) with eqn. (19-9) it can be concluded that the maximum circumferential stress in a rotating disc is twice as large, when there is a small hole at its axis of rotation, as that when the disc is solid.
Also, when \( r_1 \) approaches \( r_2 \), such that \( r_1 = r_2 = r \), we get

\[
(\sigma_e)_{\text{max}} = \rho \omega^2 r^2
\]  

...(19.17)

which is the same as equation (19-1) obtained for the case of a ring.

**Example 19.3.** Determine the intensities of principal stresses in flat steel disc of uniform thickness having a diameter of 1 m and rotating at 2400 r.p.m.

What will be the stresses if the disc has a central hole of 0.2m diameter?

Take Poisson's ratio = 1/3, and \( \rho = 7850 \text{ kg/m}^3 \).

**Solution.** Density of material, \( \rho = 7850 \text{ kg/m}^3 \)

Poisson's ratio,

\[
\frac{1}{m} = \frac{1}{3}
\]

\( N = 2400 \text{ r.p.m.} \)

\[\therefore\] Angular speed,

\[
\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 80 \pi \text{ rad./s}
\]

**Case 1. Solid disc :**

Radius of the disc,

\( r = 0.5 \text{ m} \)

The intensities of radial and circumferential (or hoop) stresses are given by :

\[
\sigma_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2 
\]  

...(1) [Eqn. (19.3)]

\[
\sigma_e = \frac{C_1}{2} - \frac{C_2}{r^2} - \left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 r^2
\]  

...(2) [Eqn. (19.4)]

Since \( \sigma_r \) cannot have infinite value at \( r = 0 \), we get

\[
C_2 = 0
\]  

...(i)

Also, at \( r = r \) (i.e., 0.5 m), \( \sigma_e = 0 \)

\[\therefore\]

\[
0 = \frac{C_1}{2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]

\[
\therefore \quad C_1 = 2 \times \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 r^2
\]  

...(ii)

Now,

\[
\left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 = \left( \frac{3 + \frac{1}{m}}{8} \right) \times 7850 \times (80 \pi)^2 = 206.6 \times 10^6 \text{ N/m}^4
\]

\[\therefore\]

\[
C_1 = 2 \times 206.6 \times 10^6 \times (0.5)^2 \times 10^{-6} \text{ MN/m}^2 = 103.3 \text{ MN/m}^2
\]

Also,

\[
\left( \frac{1 + \frac{3}{m}}{8} \right) \rho \omega^2 = \left( \frac{1+3 \times 1/3}{8} \right) \times 7850 \times (80 \pi)^2 = 123.9 \times 10^6 \text{ N/m}^4
\]

Substituting the values in (1) and (2), we get

\[
\sigma_r = 51.65 - 206.6 \times 10^6 \times r^2 \times 10^{-6} \text{ MN/m}^2
\]  

...(3)

and,

\[
\sigma_e = 51.65 - 123.9 \times 10^6 r^2 \times 10^{-6} \text{ MN/m}^2
\]  

...(4)

At \( r = 0 \),

\[
\sigma_r = \sigma_e = 51.65 \text{ MN/m}^2 \quad (\text{Ans.})
\]

At \( r = 0.5 \text{ m} \),

\[
\sigma_r = 51.65 - 206.6 \times 10^6 \times (0.5)^2 \times 10^{-6} = 0
\]

and,

\[
\sigma_e = 51.65 - 123.9 \times 10^6 \times (0.5)^2 \times 10^{-6} = 20.67 \text{ MN/m}^2 \quad (\text{Ans.})
\]

It may be noted that the above values of \( \sigma_r \) and \( \sigma_e \) are also the principal stresses.
Case II. Disc with a central hole:

Here, \( r_1 = 0.1 \text{ m}, \ r_2 = 0.5 \text{ m} \)

At \( r = 0.1 \text{ m}, \ \sigma_r = 0 \) \[ 0 = \frac{C_1}{2} + \frac{C_2}{0.4^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \times 0.1^2 \]

\[ C_1 + 200C_2 = 4.132 \] \[ \ldots (iii) \]

At \( r = 0.5 \text{ m}, \ \sigma_r = 0 \) \[ 0 = \frac{C_1}{2} + \frac{C_2}{0.5^2} - \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 \times 0.5^2 \]

\[ C_1 + 8C_2 = 103.3 \] \[ \ldots (iv) \]

From (iii) and (iv), we get

\[ C_1 = 107.43, \ C_2 = -0.516 \]

Hence the stresses are given by:

\[ \sigma_r = 53.7 - \frac{0.516}{r^2} - 206.6 \times 10^6 \times r^2 \times 10^{-6} \] \[ \ldots (5) \]

and

\[ \sigma_c = 53.7 + \frac{0.516}{r^2} - 123.9 \times 10^6 \times r^2 \times 10^{-6} \] \[ \ldots (6) \]

At \( r = 0.1 \text{ m}, \)

\[ \sigma_c = \left( \sigma_c \right)_{\text{max}} = 53.7 + \frac{0.516}{0.1^2} - 123.9 \times 10^6 \times 0.1^2 \times 10^{-6} \]

\[ = 53.7 + 51.6 - 1.239 = 104.06 \text{ MN/m}^2 \] (Ans.)

At \( r = 0.5 \text{ m}, \)

\[ \sigma_c = 53.7 + \frac{0.516}{0.5^2} - 123.9 \times 10^6 \times 0.5^2 \times 10^{-6} \]

\[ = 53.7 + 2.064 - 30.97 = 24.79 \text{ MN/m}^2 \] (Ans.)

In hydroturbine generators, the water force turns the turbines which are connected via shafts to generators. These generators convert the torque into electric energy.
\( \sigma_r \) is maximum at

\[
\frac{r}{r_2} = \sqrt{\frac{0.1 \times 0.5}{0.2236}} = 0.2236 \text{ m}
\]

\[
(\sigma_r)_{\text{max}} = 53.7 - \frac{0.516}{(0.2236)^2} = 206.6 \times 10^6 \times (0.2236)^2 \times 10^{-6}
\]

\[
= 53.7 - 10.32 - 10.32 = 33.06 \text{ MN/m}^2 \quad \text{(Ans.)}
\]

**Example 19-4.** A steel disc of uniform thickness and of diameter 400 mm is rotating about its axis at 2000 r.p.m. The density of the material is 7700 kg/m\(^3\) and Poisson's ratio is 0.3. Determine the variations of circumferential and radial stresses.

**Solution.** Radius of the disc, \( r_2 = 200 \text{ mm} = 0.2 \text{ m} \)

Speed, \( N = 2000 \text{ r.p.m.} \)

Density of material, \( \rho = 7700 \text{ kg/m}^3 \)

Poisson's ratio, \( \nu = 0.3 \)

\[
\sigma_r = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 (r_2^2 - r^2) \quad \text{...(i)} \quad [\text{Eqn. (19.5)}]
\]

\[
\sigma_c = \frac{\rho \omega^2}{8} \left[ \left( \frac{3 + \frac{1}{m}}{r_2^2} \right) - 1 + \frac{3}{m} r^2 \right] \quad \text{...(ii)} \quad [\text{Eqn. (19.6)}]
\]

Now, \( \frac{\rho \omega^2}{8} \left( \frac{3 + \frac{1}{m}}{r_2^2} \right) = \frac{7700 \times (209.44)^2}{8} \quad (3 + 0.3) = 139.3 \times 10^6 \text{ N/m}^2 \)

\[
\left[ \text{where, } \omega = \frac{2 \pi N}{60} = \frac{2 \pi \times 2000}{60} = 209.4 \text{ rad/s} \right]
\]

and, \( \frac{\rho \omega^2}{8} \left( 1 + \frac{3}{m} \right) = \frac{7700 \times (209.44)^2}{8} (1 + 3 \times 0.3) = 80.22 \times 10^6 \text{ N/m}^2 \)

**Radial stress (\( \sigma_r \))**

At \( r = 0 \), \( \sigma_r = 139.3 \times 10^6 (0 - 0.2^2) = 5.57 \text{ MN/m}^2 \) \( \text{(i.e., at the centre)} \)

At \( r = 0.05 \text{ m} \), \( \sigma_r = 139.3 \times 10^6 (0.02^2 - 0.05^2) \times 10^{-6} = 5.22 \text{ MN/m}^2 \)

At \( r = 0.1 \text{ m} \), \( \sigma_r = 139.3 \times 10^6 (0.2^2 - 0.1^2) \times 10^{-6} = 4.18 \text{ MN/m}^2 \)

At \( r = 0.15 \text{ m} \), \( \sigma_r = 139.3 \times 10^6 (0.2^2 - 0.15^2) \times 10^{-6} = 2.44 \text{ MN/m}^2 \)

At \( r = 0.2 \text{ m} \), \( \sigma_r = 139.3 \times 10^6 (0.2^2 - 0.2^2) \times 10^{-6} = 0 \)

**Circumferential stress (\( \sigma_c \))**

\[
\sigma_c = \frac{\rho \omega^2}{8} \left( \frac{3 + \frac{1}{m}}{r_2^2} \right) - \frac{\rho \omega^2}{8} \left( 1 + \frac{3}{m} \right) r^2
\]

\[
= 139.3 \times 10^6 \times 0.22 \times 10^{-6} - 80.22 \times 10^6 \times 0.22 \times 10^{-6} \times 10^{-6} \times \text{MN/m}^2
\]

\[
\text{or,} \quad \sigma_c = 5.57 - 80.22 \times 0.2^2 \text{ MN/m}^2
\]

At \( r = 0 \), \( \sigma_c = 0 \) \( \text{(i.e., at the centre)} \)

At \( r = 0.05 \text{ m} \), \( \sigma_c = 5.57 - 0 = 5.57 \text{ MN/m}^2 \)

At \( r = 0.1 \text{ m} \), \( \sigma_c = 5.57 - 80.22 \times 0.05^2 = 5.37 \text{ MN/m}^2 \)

At \( r = 0.15 \text{ m} \), \( \sigma_c = 5.57 - 80.22 \times 0.1^2 = 4.77 \text{ MN/m}^2 \)

At \( r = 0.2 \text{ m} \), \( \sigma_c = 5.57 - 80.22 \times 0.2^2 = 2.36 \text{ MN/m}^2 \)
Stress variations:
The variations of circumferential and radial stresses along the radius of the disc is shown in Fig. 19.6.

Example 19-5. A disc of uniform thickness having inner and outer diameters 100 mm and 400 mm respectively is rotating at 5000 r.p.m. about its axis. The density of the material of the disc is 7800 kg/m³ and Poisson’s ratio is 0-28.

Determine the stress variations along the radius of the disc.

Solution. Inner radius of the disc, \( r_1 \)
\[
= \frac{100}{2} = 50 \text{ mm} = 0.05 \text{ m}
\]

Outer radius of the disc, \( r_2 \)
\[
= \frac{400}{2} = 200 \text{ mm} = 0.2 \text{ m}
\]

Speed of the disc, \( N = 5000 \text{ r.p.m.} \)

\[
\therefore \text{Angular speed, } \omega = \frac{2\pi \times 5000}{60} = 523.6 \text{ rad/s}
\]

(ii) Principal stresses, maximum shear stress:
\[
\sigma_r = \left(3 + \frac{1}{m}\right) \rho \omega^2 \left[\frac{r_1^2 + r_2^2}{r^2} - \frac{r_1^2 r_2^2}{r^2} - r^2\right] \quad \text{[Eqn. (19-10)]}
\]

or,
\[
\sigma_r = \left(3 + \frac{1}{m}\right) \rho \omega^2 [r_1^2 + r_2^2]
\]

\[
- \left(3 + \frac{1}{m}\right) \rho \omega^2 \frac{r_1^2 r_2^2}{r^2} = \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2
\]

\[
\sigma_\theta = \left(3 + \frac{1}{m}\right) \rho \omega^2 \left[\frac{r_1^2 + r_2^2}{r^2} + \frac{r_1^2 r_2^2}{r^2} - \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 \right] \quad \text{[Eqn. (19-11)]}
\]

and,
\[
\sigma_r = \left(3 + \frac{1}{m}\right) \rho \omega^2 \left[\frac{r_1^2 + r_2^2}{r^2} + \frac{r_1^2 r_2^2}{r^2} - \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 \right]
\]

or,
\[
\sigma_\theta = \left(3 + \frac{1}{m}\right) \rho \omega^2 \left[\frac{r_1^2 + r_2^2}{r^2} + \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 - \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 \right]
\]

\[
\sigma_\theta = \left(3 + \frac{1}{m}\right) \rho \omega^2 \left[\frac{r_1^2 + r_2^2}{r^2} + \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 - \left(3 + \frac{1}{m}\right) \rho \omega^2 r_2^2 \right]
\]
We have:
\[
\left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 = \left(\frac{3 + 0.28}{8}\right) \times 7800 \times (523.6)^2 \times 10^{-6} \text{ MN/m}^4 = 876.75 \text{ MN/m}^4
\]
\[
\left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left(r_1^2 + r_2^2\right) = 876.75 \times (0.05^2 + 0.2^2) = 37.26 \text{ MN/m}^2
\]
\[
\left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r_1^2 r_2^2 = 876.75 \times 0.05^2 \times 0.2^2 = 0.0876 \text{ MN}
\]
\[
\left(\frac{1 + \frac{3}{m}}{8}\right) \rho \omega^2 = \left(\frac{1 + 3 \times 0.28}{8}\right) \times 7800 \times (523.6)^2 \times 10^{-6} \text{ MN/m}^4 = 491.8 \text{ MN/m}^4
\]

**Radial stress, \( \sigma_r \):**
\[
\sigma_r = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left(r^2 + r_2^2\right) - \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \frac{r_1^2 r_2^2}{r^2} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r^2 \quad \text{[Eqn. (1)]}
\]

or,
\[
\sigma_r = 37.26 - \frac{0.0876}{r^2} - 876.75 \frac{r^2}{r^2} = 37.26 - 876.75
\]

At \( r = 0.05 \text{ m} \),
\[
\sigma_r = 37.26 - 0.0876 \frac{0.05^2}{0.05^2} - 876.75 \times 0.05^2 = 37.26 - 35.06 - 2.20 = 0
\]

At \( r = 0.1 \text{ m} \),
\[
\sigma_r = 37.26 - \frac{0.0876}{(0.1)^2} - 876.75 \times 0.1^2 = 37.26 - 8.76 - 8.76 = 19.74 \text{ MN/m}^2
\]

At \( r = 0.15 \text{ m} \),
\[
\sigma_r = 37.26 - \frac{0.0876}{0.15^2} - 876.75 \times 0.15^2 = 37.26 - 3.89 - 19.73 = 13.64 \text{ MN/m}^2
\]

At \( r = 0.2 \text{ m} \),
\[
\sigma_r = 37.26 - \frac{0.0876}{0.2^2} - 876.75 \times 0.2^2 = 37.26 - 2.19 - 35.07 = 0
\]

Maximum \( \sigma_r \) occurs at \( r = \sqrt{r_1 r_2} = \sqrt{0.05^2 \times 0.2^2} = 0.1 \text{ m} \)

* i.e., \( \sigma_{r/\text{max}} = 19.74 \text{ MN/m}^2 *

**Circumferential stress, \( \sigma_c \):**
\[
\sigma_c = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left(r^2 + r_2^2\right) + \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \frac{r_1^2 r_2^2}{r^2} - \left(\frac{1 + \frac{3}{m}}{8}\right) \rho \omega^2 r^2 \quad \text{[Eqn. (2)]}
\]
\[
= 37.26 - \frac{0.0876}{r^2} - 491.8 \frac{r^2}{r^2}
\]
At $r = 0.05 \text{ m}$,

$$\sigma_c = 37.26 + \frac{0.0876}{(0.05)^2} - 491.8 \times 0.05^2$$

$$= 37.26 + 35.06 - 1.23 = 71.09 \text{ MN/m}^2$$

At $r = 0.1 \text{ m}$,

$$\sigma_c = 37.26 + \frac{0.0876}{(0.1)^2} - 491.8 \times 0.1^2$$

$$= 37.26 + 8.76 - 4.92 = 41.11 \text{ MN/m}^2$$

Proper fit of tyre and wheels is essential to safely distribute the torque and shear force.
At $r = 0.15$ m,
\[
\sigma_c = 37.26 + \frac{0.0876}{0.15^2} - 491.8 \times 0.15^2
\]
\[
= 37.26 + 3.89 - 11.06 = 30.09 \text{ MN/m}^2
\]
At $r = 0.2$ m,
\[
\sigma_c = 37.26 + \frac{0.0876}{0.2^2} - 491.8 \times 0.2^2
\]
\[
= 37.26 + 2.19 - 19.67 = 19.78 \text{ MN/m}^2
\]
Maximum stress occurs at the inner radius, where circumferential stress $\sigma_c$ is maximum.
\[\therefore \text{Maximum principal stress} = 71.09 \text{ MN/m}^2 \quad \text{(Ans.)}\]
Again, at the inner radius the principal stresses are 71.09 MN/m², 0, 0
\[\therefore \text{Maximum shear stress,} \quad \tau_{\text{max}} = \frac{71.09}{2} = 35.54 \text{ MN/m}^2 \quad \text{(Ans.)}\]
Fig. 19.7 shows the stress variation along the radius of the disc.

19.4. DISC OF UNIFORM STRENGTH

A disc of uniform strength is the one in which the values of radial and circumferential stresses are equal in magnitude for all values of radius $r$. This means that the disc of uniform strength must have a varying thickness.

In industry there are several components such as rotor of a steam turbine which have constant strength throughout the radius and are designed by varying their thickness.

Fig. 19.8(a) shows the elevation of a disc of uniform strength and Fig. 19.8(b) shows the free body diagram of an element $ABCD$ of the disc which subtends an angle $d\theta$ at the centre $O$. Further the element is bounded by surfaces $DC$ and $AB$ whose radii are $r$ and $(r + dr)$ respectively and corresponding axial thicknesses as $t$ and $(t + dt)$.

![Diagram](image)

Let, $\sigma =$ Uniform stress in the radial and circumferential directions.
Volume of the element = $r \, d\theta \cdot t \cdot dr$
Centrifugal force acting on the element $ABCD$ due to rotation
\[= \rho \cdot r \, d\theta \cdot t \, dr \cdot \omega^2 r = \rho \, d\theta \cdot t \, dr \cdot \omega^2 r^2\]
Radial force on the face $DC = r \, d\theta$. t. $\sigma$
Radial force on the face $AB = (r + dr) \, d\theta$. $(t + dt) \, \sigma$
Circumferential force on faces $BC$ and $DA = t. \, dr. \, \sigma$

(inclined at an angle $\frac{d\theta}{2}$ to the radial direction)

Resolving all the forces along the radial direction and considering equilibrium, we get

$$\rho \, d\theta. \, t. \, dr. \, \omega^2 r^2 + (r + dr) \, d\theta. (t + dt) \, \sigma = r \, d\theta. \, t. \, \sigma + 2 \, t. \, dr. \, \sigma \sin \frac{d\theta}{2} \cdot \sigma$$

...(1)

Cancelling $d\theta$ on both sides, eqn. (1) is simplified as follows:

$$\rho \, t. \, dr. \, \omega^2 r^2 + rt \, \sigma + r \, dt \, \sigma + dr. \, t. \, \sigma + dr. \, dt. \, \sigma = r. t. \, \sigma + t. \, dr. \, \sigma$$

Neglecting the product of small quantities, we get

$$\rho \, t. \, dr. \, \omega^2 r^2 + r. \, dt. \, \sigma = 0$$

or,

$$\frac{dt}{t} = -\frac{\rho \, \omega^2 \, r}{\sigma} \, dr$$

Integrating both sides, we get

$$\log e \, t = \frac{\rho \, \omega^2 \, r^2}{2\sigma} + \log e \, A$$

(where, $\log e \, A$ is a constant of integration)

$$\log e \, \frac{t}{A} = -\frac{\rho \, \omega^2 \, r^2}{2\sigma}$$

$$\frac{t}{A} = e^{-\frac{\rho \, \omega^2 \, r^2}{2\sigma}}$$

At,

$$r = 0, \, t = t_0$$
$$t_0 = A$$

$\therefore$ Thickness at any radius, $t = t_0 \, e^{-\frac{\rho \, \omega^2 \, r^2}{2\sigma}}$ ... (19-18)

**Example 19-6.** A steam turbine rotor is to be designed so that the radial and circumferential stresses are to be the same and constant throughout and equal to 90 MN/m$^2$, when running at 4000 r.p.m. If the axial thickness at the centre is 20 mm, what is the thickness at a radius of 400 mm? The same density of material of the rotor is 7800 kg/m$^3$.

**Sol.**

Allowable stress, $\sigma = 90$ MN/m$^2$

Speed of the rotor, $N = 4000$ r.p.m.

$\therefore$ Angular speed,$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 4000}{60} = 418.88$ rad./s.

Axial thickness at the centre, $t_0 = 20$ mm

Density of material, $\rho = 7800$ kg/m$^3$
19.5. ROTATING LONG CYLINDERS

The analysis of a rotating long cylinder is similar to that of a thin disc, the only difference being that the length of the cylinder along the axis is large as compared to the radius and axial stress is considered along the length of the cylinder.

While developing theory for rotating long cylinders following assumptions are made:

1. Even at high speeds of rotation plane cross-sections remain plane. Strictly speaking this is true for sections far away from the ends.
2. At the central cross-sectional plane of the cylinder, due to symmetry, shear stress is zero and thus the radial, circumferential and longitudinal stresses will be principal stresses. It will be assumed that it is nearly true at all sections except near the ends.

Let,

\( \sigma_r \) = Radial stress,

\( \sigma_c \) = Circumferential (or hoop) stress, and

\( \sigma_l \) = Longitudinal (or axial) stress.

Let these stresses \((\sigma_r, \sigma_c, \sigma_l)\) act on any element of a section of the cylinder of radius \(r\) [Fig. 19.3 (b)]

Then, radial strain,

\[
\varepsilon_r = \frac{1}{E} \left[ \sigma_r - \frac{1}{m} (\sigma_c + \sigma_l) \right] = \frac{du}{dr} \tag{i}
\]

Circumferential strain,

\[
\varepsilon_c = \frac{1}{E} \left[ \sigma_c - \frac{1}{m} (\sigma_r + \sigma_l) \right] = \frac{u}{r} \tag{ii}
\]

Longitudinal strain,

\[
\varepsilon_l = \frac{1}{E} \left[ \sigma_l - \frac{1}{m} (\sigma_r + \sigma_c) \right] \tag{iii}
\]

(where, \(E\) = Young’s modulus)

From eqn. (ii), we have

\[
Eu = r \left[ \sigma_c - \frac{1}{m} (\sigma_r + \sigma_l) \right]
\]

Differentiating w.r.t. \(r\), we get

\[
E \frac{du}{dr} = \left[ \sigma_c - \frac{1}{m} (\sigma_r + \sigma_l) \right] + r \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \left( \frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr} \right) \right]
\]

\[
= \sigma_r - \frac{1}{m} (\sigma_c + \sigma_l)
\]

[From eqn. (i)]

\[
\therefore \quad (\sigma_r - \sigma_c) \left( 1 + \frac{1}{m} \right) = r \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \left( \frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr} \right) \right]
\]

\tag{iv}
From eqn. (iii), we have
\[ Ee_l = \sigma_l - \frac{1}{m} (\sigma_r + \sigma_c) = \text{constant} = C_1 \]
\[ \therefore \]
\[ \sigma_l = C_1 + \frac{1}{m} (\sigma_r + \sigma_c) \]
Differentiating w.r.t. \( r \), we get
\[ \frac{d\sigma_l}{dr} = \frac{1}{m} \left[ \frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} \right] \]
Substituting in eqn. (iv), we get
\[ (\sigma_r - \sigma_c) \left( 1 + \frac{1}{m} \right) = r \left[ \frac{d\sigma_c}{dr} - \frac{1}{m} \left( \frac{d\sigma_r}{dr} + \frac{1}{m} \left( \frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} \right) \right) \right] \]
\[ = r \left[ \left( 1 - \left( \frac{1}{m} \right)^2 \right) \frac{d\sigma_c}{dr} - \frac{1}{m} \left( 1 + \frac{1}{m} \right) \frac{d\sigma_r}{dr} \right] \]
or
\[ \sigma_r - \sigma_c = r \left[ \left( 1 - \frac{1}{m} \right) \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] \quad \ldots \text{(v)} \]
Also considering the equilibrium of an element of the cylinder between angular position \( \theta \) and \( \theta + d\theta \) and radii \( r \) and \( r + dr \), we can get as in the case of a rotating disc
\[ \sigma_r - \sigma_c = - \left( r \frac{d\sigma_r}{dr} + \rho \omega^2 r^2 \right) \quad \ldots \text{(vi)} \]
Comparing eqns. (v) and (vi), we get
\[ r \left[ \left( 1 - \frac{1}{m} \right) \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} \right] = - r \left( \frac{d\sigma_r}{dr} + \rho \omega^2 r \right) \]
\[ \left( 1 - \frac{1}{m} \right) \frac{d\sigma_c}{dr} - \frac{1}{m} \frac{d\sigma_r}{dr} = - \left( \frac{d\sigma_r}{dr} + \rho \omega^2 r \right) \]
\[ \left( 1 - \frac{1}{m} \right) \frac{d\sigma_c}{dr} - \left( 1 - \frac{1}{m} \right) \frac{d\sigma_c}{dr} = - \rho \omega^2 r \]
\[ \left( 1 - \frac{1}{m} \right) \frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} = - \rho \omega^2 r \]
\[ \frac{d\sigma_r}{dr} + \frac{d\sigma_c}{dr} = - \frac{\rho}{\left( 1 - \frac{1}{m} \right)} \omega^2 r \]
\[ \frac{d}{dr} (\sigma_r + \sigma_c) = - \frac{\rho}{\left( 1 - \frac{1}{m} \right)} \omega^2 r \quad \ldots \text{(vii)} \]
Integrating both sides, we get
\[ \sigma_r + \sigma_c = - \frac{\rho}{\left( 1 - \frac{1}{m} \right)} \omega^2 \cdot \frac{r^2}{2} + C_2 \]
Adding eqns. (vi) and (vii), we get

\[ 2\sigma_r = - \left[ \frac{d\sigma_r}{dr} + \rho \cdot \omega^2 \cdot r^2 \right] - \left( \frac{\rho}{1 - \frac{1}{m}} \right) \cdot \frac{\omega^2 r^2}{2} + C_2 \]

\[ 2\sigma_r + \frac{d\sigma_r}{dr} = - \rho \cdot \frac{\omega^2 r^2}{2} \left[ \frac{3 - \frac{2}{m}}{2 \left( 1 - \frac{1}{m} \right)} \right] + C_2 \]

Multiplying both sides by \( r \), we get

\[ 2r\sigma_r + r^2 \frac{d\sigma_r}{dr} = - \rho \cdot \frac{\omega^2 r^3}{2} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] + r C_2 \]

\[ \frac{d}{dr} (r^2 \sigma_r) = - \rho \cdot \frac{\omega^2 r^3}{2} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] + r C_2 \]

Integrating both sides, we get

\[ r^2 \sigma_r = - \rho \cdot \frac{\omega^2 r^4}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] + \frac{r^2}{2} C_2 + C_3 \]

\[ \sigma_r = - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] + \frac{C_2}{2} + \frac{C_3}{r^2} \]

or,

\[ \sigma_r = \frac{C_2}{2} + \frac{C_3}{r^2} - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \]

Substituting in equation (vii), we get

\[ \sigma_e = - \frac{\rho}{1 - \frac{1}{m}} \cdot \frac{\omega^2 r^2}{2} + C_2 - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] + \frac{C_2}{2} + \frac{C_3}{r^2} \]

\[ = - \frac{\rho}{1 - \frac{1}{m}} \cdot \frac{\omega^2 r^2}{2} + C_2 + \rho \cdot \frac{\omega^2 r^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) - \frac{C_2}{2} - \frac{C_3}{r^2} \]
\[
\begin{align*}
\sigma_r &= \frac{C_2}{2} - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \\
\sigma_\theta &= \frac{C_2}{2} - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{1 + \frac{2}{m}}{1 - \frac{1}{m}} \right]
\end{align*}
\]

Equations (19.19) and (19.20) are the governing equations for a rotating cylinder in which \(C_2\) and \(C_3\) (constants of integration) are evaluated with the help of end conditions.

19.5.1. Solid Cylinder

Since the stresses cannot be infinite at the centre, therefore \(C_3 = 0\). The expressions for stresses will now be as follows:

\[
\begin{align*}
\sigma_r &= \frac{C_2}{2} - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \\
\sigma_\theta &= \frac{C_2}{2} - \rho \cdot \frac{\omega^2 r^2}{8} \left[ \frac{1 + \frac{2}{m}}{1 - \frac{1}{m}} \right]
\end{align*}
\]

For a solid cylinder with a free surface, at \(r = r_2\), \(\sigma_r = 0\)

Cylinder boring on a lathe. Most machining processes are performed with rotary motion.
\[ \sigma_r = \frac{\rho \omega^2 (r^2 - r_0^2)}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \]

\[ \sigma_c = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left( r_2^2 - \left( 1 + \frac{2}{m} \right) r_0^2 \right) \]

The maximum stress occurs at the centre of the cylinder, where \( r = 0 \)

\[ (\sigma_r)_{\text{max}} = (\sigma_c)_{\text{max}} = \rho \frac{\omega^2 r_2^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \]

19-5.2. Hollow cylinder

\[ \sigma_r = \frac{C_2}{2} + \frac{C_3}{r^2} - \rho \frac{\omega^2 r^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \]  

[Eqn. (19-19)]

At \( r = r_1, \sigma_r = 0 \)

and, \( r = r_2, \sigma_r = 0 \)

\[ 0 = \frac{C_2}{2} + \frac{C_3}{r_1^2} - \rho \frac{\omega^2 r_1^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \]

and,

\[ 0 = \frac{C_2}{2} + \frac{C_3}{r_2^2} - \rho \frac{\omega^2 r_2^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \]

From these boundary conditions

\[ C_3 \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) - \rho \frac{\omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) (r_1^2 - r_2^2) = 0 \]

\[ C_3 \left( \frac{r_2^2 - r_1^2}{r_1^2 r_2^2} \right) = - \rho \frac{\omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) (r_2^2 - r_1^2) \]
\[ C_3 = -\frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \frac{r_1^2 r_2^2}{m} \]

Again,

\[ \frac{C_2}{2} = -\frac{C_3}{r_1^2} + \frac{\rho \cdot \omega^2 r_1^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \frac{r_2}{m} \]

\[ = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \frac{r_2}{m} + \frac{\rho \cdot \omega^2 r_1^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \frac{r_1^2 + r_2^2}{m} \]

or,

\[ \frac{C_2}{2} = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) (r_1^2 + r_2^2) \]

The expressions for stresses will now be:

Radial stress:

\[ \sigma_r = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \left[ (r_1^2 + r_2^2) - \frac{r_1^2 r_2^2}{r^2} - r^2 \right] \]

...(19-24)

[substituting the values of \( C_2 \) and \( C_3 \) in eqn. (19-19)]

For \( \sigma_r \) to be maximum, \( \frac{d\sigma_r}{dr} = 0 \)

\[ \frac{2r_1^2 r_2^2}{r^3} - 2r = 0 \]

\[ r^4 = r_1^2 r_2^2 \]

\[ r = \sqrt{r_1 \cdot r_2} \]

\[ (\sigma_r)_{\text{max}} = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \left[ (r_1^2 + r_2^2) - \frac{r_1^2 r_2^2}{r_1 r_2} - r_1 r_2 \right] \]

\[ = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) [r_1^2 + r_2^2 - 2r_1 r_2] \]

\[ (\sigma_r)_{\text{max}} = \frac{\rho \omega^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) (r_1 - r_2)^2 \]

...(19-25)
Circumferential stress:
The expression for the circumferential or hoop stress will be

\[
\sigma_c = \frac{C_2}{2} - \frac{C_3}{r^2} - \frac{\rho \omega^2 r^2}{8} \left( 1 + \frac{2}{m} \right) \left( 1 - \frac{1}{m} \right)
\]

[Eqn. (19-20)]

Substituting the values of \(C_2\) and \(C_3\), we get

\[
\sigma_c = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] (r_1^2 + r_2^2) + \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \frac{r_1^2 r_2^2}{r^2} - \frac{\rho \omega^2 r^2}{8} \left[ \frac{1 + \frac{2}{m}}{1 - \frac{1}{m}} \right]
\]

or,

\[
\sigma_c = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left( r_1^2 + r_2^2 \right) + \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \frac{r_1^2 r_2^2}{r^2} - \frac{\rho \omega^2 r^2}{8} \left[ \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right] r^2
\]

\[\sigma_c\] is maximum at \( r = r_1 \)

\[
(\sigma_c)_{\text{max}} = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left( 2r_1^2 + r_1^2 \right) - \frac{\rho \omega^2}{8} \left[ \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right] r_1^2
\]

\[\text{Example 19-7.}\] A long cylinder of 300 mm radius is rotating at 4500 r.p.m. The density of material is 7800 kg/m\(^3\) and Poisson's ratio is 0.3.

(i) Calculate the maximum stress in the cylinder;

(ii) Draw the variations of radial and circumferential stresses along the radius.

**Solution.** Radius of the cylinder, \( r_2 = 300 \text{ mm} = 0.3 \text{ m} \)

Speed of the rotor, \( N = 4500 \text{ r.p.m.} \)

\[
\omega = \frac{2\pi N}{60} \text{ rad./s} = \frac{2\pi \times 4500}{60} = 471.2 \text{ rad/s}
\]

(i) Maximum stress in the cylinder:

\[
\frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} = \frac{3 - 2 \times 0.3}{1 - 0.3} = \frac{2.4}{0.7} = 3.428
\]

\[
\frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} = \frac{1 + 2 \times 0.3}{3 - 2 \times 0.3} = \frac{1.6}{2.4} = 0.667
\]

Maximum radial and circumferential stresses occur at the centre, where \( r = 0 \)
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\[
(\sigma_r)_{\text{max}} = (\sigma_c)_{\text{max}} = \frac{\rho \omega^2 r_1^2}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right)
\]

...[Eqn. (19-23)]

\[
= \frac{7800 \times (471.2)^2 \times (0.3)^2}{8} \times 3428 \times 10^{-6} \text{ MN/m}^2
\]

\[
= 66.79 \text{ MN/m}^2 \quad \text{(Ans.)}
\]

(ii) Variations of stresses:

Radial stress \(\sigma_r\):

\[
\sigma_r = \frac{\rho \omega^2 (r_2^2 - r_1^2)}{8} \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right)
\]

...[Eqn. (19-21)]

\[
= \frac{7800 \times (471.2)^2 \times (0.3^2 - r^2)}{8} \times 3428 \times 10^{-6} \text{ MN/m}^2
\]

or,

\[
\sigma_r = 742.09 \ (0.09 - r^2)
\]

At \(r = 0\), \(\sigma_r = 66.79 \text{ MN/m}^2\)
At \(r = 0.06\ m\), \(\sigma_r = 64.11 \text{ MN/m}^2\)
At \(r = 0.12\ m\), \(\sigma_r = 56.1 \text{ MN/m}^2\)
At \(r = 0.18\ m\), \(\sigma_r = 42.74 \text{ MN/m}^2\)
At \(r = 0.24\ m\), \(\sigma_r = 24.04 \text{ MN/m}^2\)
At \(r = 0.3\ m\), \(\sigma_r = 0\)

Circumferential stress, \(\sigma_c\):

\[
\sigma_c = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left( r_2^2 - \left( \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right) r_1^2 \right)
\]

...[Eqn. (19-22)]

\[
= \left[ \frac{7800 \times (471.2)^2}{8} \times 3428 \right] [(0.3)^2 - 0.667 r^2] \times 10^{-6} \text{ MN/m}^2
\]

or,

\[
\sigma_c = 742.09 \ (0.09 - 0.667 r^2) \text{ MN/m}^2
\]

At \(r = 0\), \(\sigma_c = 66.79 \text{ MN/m}^2\)
At \(r = 0.06\ m\), \(\sigma_c = 65 \text{ MN/m}^2\)
At \(r = 0.12\ m\), \(\sigma_c = 59.66 \text{ MN/m}^2\)
At \(r = 0.18\ m\), \(\sigma_c = 50.75 \text{ MN/m}^2\)
At \(r = 0.24\ m\), \(\sigma_c = 38.28 \text{ MN/m}^2\)
At \(r = 0.3\ m\), \(\sigma_c = 22.24 \text{ MN/m}^2\)

Fig. 19-9 shows the variations of radial and circumferential stresses along the radius of the long cylinder.

Conveyor belt used in material handling is turned by drum conveyor pulley.
Example 19.8. A hollow cylinder, 200 mm external radius and 100 mm internal radius is rotating at 3000 r.p.m. The density of material is 7800 kg/m$^3$ and Poisson’s ratio is 0.3.

(i) Calculate the maximum stress in the cylinder;
(ii) Draw the variations of radial and hoop stresses in the cylinder.

**Solution.** Inner radius of the cylinder, $r_1 = 100$ mm = 0.1 m
External radius of the cylinder, $r_2 = 200$ mm = 0.2 m
Speed of the cylinder, $N = 3000$ r.p.m.

\[ \omega = \frac{2\pi N}{60} = \frac{2\pi \times 3000}{60} = 100\pi \text{ rad/s} \]

Density of material, $\rho = 7800$ kg/m$^3$

Poisson’s ratio, $\nu = 0.3$

(i) **Maximum stress in the cylinder.**

Maximum stress occurs at the inner radius of the cylinder.

\[
\left(\sigma_c\right)_{\text{max}} = \frac{\rho \omega^2}{8} \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left(2r_2^2 + r_1^2\right) \left(1 + \frac{2}{m}\right) - \left(\frac{3 - \frac{2}{m}}{3 - \frac{2}{m}}\right) r_1^2 \]

\[= \left[\frac{3 - 2 \times 0.3}{1 - 0.3}\right] \left(2 \times 0.2^2 + 0.1^2\right) \left(1 + 2 \times 0.3\right) - \left(\frac{3 - 2 \times 0.3}{3 - 2 \times 0.3}\right) 0.1^2 \]

\[= \left[\frac{3 - 0.6}{0.7}\right] \left(0.04 + 0.01\right) \left(1 + 0.6\right) - \left(\frac{3 - 0.6}{3 - 0.6}\right) \]

\[= \left[\frac{2.4}{0.7}\right] \left(0.05\right) \left(1.6\right) = 1.76 \times 10^6 \text{ Pa} \]

\[\rho \omega^2 = \frac{7800 \times (100\pi)^2}{8} = 96.23 \times 10^6 \text{ Pa} \]
\[ (\sigma_r)_{\text{max}} = 96.23 \times 10^6 \times 3.428 \left( \frac{(2 \times 0.2^2 + 0.1^2) - 0.667 \times 0.1^2}{r^2} \right) \times 10^{-6} \text{MN/m}^2 \]

\[ = 27.5 \text{ MN/m}^2 \quad \text{(Ans.)} \]

**Radial stress \( \sigma_r \):**

\[
\sigma_r = \frac{\rho \alpha}{8} \left[ \frac{3 - \frac{2}{m}}{\frac{1}{1 - \frac{m}{1}}} \right] \left[ \frac{r_1^2 + r_2^2}{r^2} - \frac{r_1^2 \cdot r_2^2}{r^2} - r^2 \right] \quad \text{[Eqn. 19.24]}
\]

\[
= 96.23 \times 10^6 \left( 3.428 \right) \left( \frac{(0.1)^2 + (0.2)^2 - \frac{(0.1)^2 \times (0.2)^2}{r^2}}{r^2} - r^2 \right) \times 10^{-6} \text{MN/m}^2
\]

or,

\[
\sigma_r = 329.87 \left[ 0.05 + 0.0004 - \frac{0.0004}{r^2} - r^2 \right] \text{MN/m}^2
\]

or,

\[
\sigma_r = 329.87 \left[ 0.05 - \frac{0.0004}{r^2} - r^2 \right] \text{MN/m}^2
\]

At \( r = 0.1 \) m,

\[
\sigma_r = 329.87 \left( 0.05 - \frac{0.0004}{0.1^2} - 0.1^2 \right) = 0
\]

At \( r = 0.15 \) m,

\[
\sigma_r = 329.87 \left( 0.05 - \frac{0.0004}{0.15^2} - 0.15^2 \right) = 3.21 \text{ MN/m}^2
\]

At \( r = 0.2 \) m,

\[
\sigma_r = 329.87 \left( 0.05 - \frac{0.0004}{0.2^2} - 0.2^2 \right) = 0
\]

\((\sigma_r)_{\text{max}} \) occurs at \( r = \sqrt{\frac{1}{1 \cdot 2}} = \sqrt{0.4 \times 0.2} = 0.1414 \) m

\[ (\sigma_r)_{\text{max}} = 329.87 \left( 0.05 - \frac{0.0004}{0.1414^2} - 0.1414^2 \right) = 3.3 \text{ MN/m}^2 \]

**Circumferential stress, \( \sigma_c \):**

\[
\sigma_c = \frac{\rho \alpha}{8} \left[ \frac{3 - \frac{2}{m}}{\frac{1}{1 - \frac{m}{1}}} \right] \left[ \frac{r_1^2 + r_2^2}{r^2} + \frac{r_1^2 \cdot r_2^2}{r^2} - \left( \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right) r^2 \right] \quad \text{[Eqn. 19.26]}
\]

\[
= 96.23 \times 10^6 \times 3.428 \left( 0.1^2 + 0.2^2 + \frac{0.1^2 \times 0.2^2}{r^2} - 0.667 \times 0.2^2 \right) \times 10^{-6} \text{MN/m}^2
\]

or,

\[
\sigma_c = 329.87 \left( 0.05 + \frac{0.0004}{r^2} - 0.667 \times 0.2^2 \right) \text{MN/m}^2
\]

At \( r = 0.1 \) m, \( \sigma_r = 329.87 \left( 0.05 + \frac{0.0004}{0.1^2} - 0.667 \times 0.1^2 \right) = 27.5 \text{ MN/m}^2 \)

At \( r = 0.15 \) m, \( \sigma_r = 329.87 \left( 0.05 + \frac{0.0004}{0.15^2} - 0.667 \times 0.15^2 \right) = 17.41 \text{ MN/m}^2 \)

At \( r = 0.2 \) m, \( \sigma_r = 329.87 \left( 0.05 + \frac{0.0004}{0.2^2} - 0.667 \times 0.2^2 \right) = 11 \text{ MN/m}^2 \)
The variations of radial and circumferential stresses are shown in figure 19.10.

**Fig. 19.10**

High torque shunt wound drilling motor.

### HIGHLIGHTS

1. For a thin ring of mean radius \( r \) rotating at an angular speed \( \omega \), circumferential stress, \( \sigma_c = \rho \omega^2 \) where, \( \rho = \text{Density of material, kg/m}^3; \ \omega = \text{Linear velocity of the ring, m/s} \)

2. **Rotating disc**
   
   (i) *Solid disc.*

   \[
   \sigma_r = \left( \frac{3 + \frac{1}{m}}{8} \right) \rho \omega^2 (r_2^2 - r_1^2)
   \]
(\sigma_r)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r_2^2

... (ii)

\sigma_c = \frac{\rho \omega^2}{8} \left(\frac{3 + \frac{1}{m}}{r_2^2} - \left(1 + \frac{3}{m}\right) r^2\right)

... (iii)

(\sigma_c)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r_2^2

... (iv)

(ii) Hollow disc:

\sigma_r = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left[r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2\right]

... (v)

(\sigma_r)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 (r_2 - \eta)^2

... (vi)

(at \ r = \sqrt{r_1 r_2})

\sigma_c = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left(1 + \frac{3}{m}\right) \frac{1}{m}\right]

... (vii)

(\sigma_c)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 \left[r_2^2 + \frac{1 - \frac{1}{m}}{3 + \frac{1}{m}} r_1^2\right]

... (viii)

where, \ \sigma_r = \text{Radial stress},

(\sigma_r)_{max} = \text{Maximum radial stress},

\sigma_c = \text{Circumferential stress},

(\sigma_c)_{max} = \text{Maximum circumferential stress},

r_1 = \text{Inner radius},

r_2 = \text{Outer radius}, and

\omega = \text{Angular velocity}.

(iii) Hollow disc with a pinhole at the centre:

(\sigma_r)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r_2^2

... (ix)

(\sigma_c)_{max} = \left(\frac{3 + \frac{1}{m}}{8}\right) \rho \omega^2 r_2^2

... (x)
3. **Disc of uniform strength:**
Thicknes at any radius,
\[ t = t_0 e^{-\frac{\rho \omega^2 r^2}{2\sigma}} \]
where,
- \( t_0 \) = Thickness at the centre, and
- \( \sigma \) = Uniform stress in the radial and circumferential directions.

4. **Rotating long cylinders:**
   
   \( i \) **Solid cylinder.**
   \[
   \sigma_r = \rho \cdot \omega^2 \left( \frac{r_2^2 - r_1^2}{8} \right) \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right]
   \]
   \[
   \sigma_r = \rho \omega^2 \left[ \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right] \left[ \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right] r^2
   \]
   \[
   (\sigma_r)_{\text{max}} = (\sigma_c)_{\text{max}} = \rho \cdot \omega^2 r_2^2 \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right)
   \]

   \( ii \) **Hollow cylinder.**
   \[
   \sigma_r = \rho \cdot \omega^2 \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \left( r_1^2 + r_2^2 - \frac{r_1^2 r_2^2}{r^2} - r^2 \right)
   \]
   \[
   (\sigma_r)_{\text{max}} = \rho \cdot \omega^2 \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) (r_1 - r_2)^2
   \]
   \[
   \sigma_c = \rho \cdot \omega^2 \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \left[ r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right] r^2
   \]
   \[
   (\sigma_c)_{\text{max}} = \rho \cdot \omega^2 \left( \frac{3 - \frac{2}{m}}{1 - \frac{1}{m}} \right) \left[ 2r_1^2 + r_1^2 \right] - \left( \frac{1 + \frac{2}{m}}{3 - \frac{2}{m}} \right) r_1^2
   \]

**OBJECTIVE TYPE QUESTIONS**

1. A thin flat ring is rotating at a speed \( v \). The circumferential stress induced is given by
   
   \( a \) \( rv \)
   
   \( b \) \( rv^2 \)
   
   \( c \) \( \frac{1}{2} \rho v^2 \)
   
   \( d \) \( \frac{1}{2} \rho v^3 \)
   
   (where, \( \rho = \) density of material)

2. In case of a solid rotating circular disc the radial stress is maximum at
   
   \( a \) the mean radius
   
   \( b \) the outer radius
   
   \( c \) square root of the radius
   
   \( d \) the centre.

3. The circumferential stress in a solid rotating disc is maximum at
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4. The radial stress in a hollow circular rotating disc is at
   (a) the inner radius
   (b) the outer radius
   (c) the mean radius
   (d) the geometric mean radius.

5. The circumferential stress in a hollow circular rotating disc is at
   (a) the inner radius
   (b) the outer radius
   (c) the mean radius
   (d) the geometric mean radius.

6. The ratio of the maximum circumferential stress in a circular rotating disc having a very small hole at the centre to the maximum circumferential stress in a solid circular rotating disc is
   (a) 1.2
   (b) 1.5
   (c) 2
   (d) 1.5

7. In case of rotating disc of uniform strength which of the following statements is correct?
   (a) Circumferential stress is constant.
   (b) Radial stress is constant.
   (c) Circumferential and radial stresses are equal to each other and are constant.
   (d) None of the above.

8. The circumferential stress in a rotating solid circular cylinder is maximum at
   (a) the mean radius
   (b) the centre
   (c) the outer radius
   (d) the square root of the outer radius.

9. The radial stress in a rotating hollow circular cylinder is maximum at
   (a) the inner radius
   (b) the outer radius
   (c) the mean radius
   (d) the geometric mean radius.

ANSWERS

1. (b)
2. (d)
3. (c)
4. (d)
5. (a)
6. (c)
7. (c)
8. (b)
9. (d).

Electric motor with shaft and bearings.